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Graded algebras associated with indecomposable vector bundles over an elliptic curve

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§1. Introduction

Let X be an elliptic curve over an algebraically closed field k with $\mathrm{char}(k) \neq 2$. Our object is to compute the graded algebra

$$\bigoplus_{i\geq 0} \mathrm{Hom}(\mathcal{E}, \mathcal{E}\otimes\mathcal{L}^{\otimes i})$$

for a line bundle $\mathcal{L}$ and a vector bundle $\mathcal{E}$ over X defined as follows. Choose a point $P \in X$ and let $\mathcal{L} = \mathcal{L}(P)$ be the line bundle associated to the divisor P. Vector bundles over X were classified by Atiyah [1]. Among them we choose the following ones. For each positive integer n there exists uniquely an indecomposable vector bundle $\mathcal{E}_n$ of rank n which is a successive extension of the trivial bundle. That is,

$$\mathcal{O}_X = \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots$$
$$0 \to \mathcal{E}_{n-1} \hookrightarrow \mathcal{E}_n \to \mathcal{O}_X \to 0 \qquad \text{exact, non split.}$$

Now put

$$\Lambda(n) = \bigoplus_{i\geq 0} \Gamma(X, \mathcal{E}nd(\mathcal{E}_n)\otimes\mathcal{L}^{\otimes i}) = \bigoplus_{i\geq 0} \mathrm{Hom}(\mathcal{E}_n, \mathcal{E}_n\otimes\mathcal{L}^{\otimes i}).$$

We aim to give an explicit description of the algebra $\Lambda(n)$.

§2. Homogeneous coordinate ring

First of all, we look at the algebra

$$S = \bigoplus_{i\geq 0} \Gamma(X, \mathcal{L}^{\otimes i}).$$

We know the following presentation of S [2, p. 336].

generators:	$t \in S_1, \quad x \in S_2, \quad y \in S_3$
relation:	$y^2 = x(x-t^2)(x-\lambda t^2) \quad \text{with } \lambda \in k - \{0,1\}.$

Also we have $S_0 = k$, $\dim S_i = i$ for $i > 0$ and a k-basis of S is given by $t^i x^j$, $t^i x^j y$ for $i, j \geq 0$. In addition, X is determined by λ as

$$\begin{aligned} X &\cong \{x_1^2 x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2)\} \subset \mathbb{P}^2 \\ P &\leftrightarrow (0:1:0) \end{aligned}$$

We fix t, x, y, λ throughout.

§3. First properties of $\Lambda(n)$

We collect here some properties of $\Lambda(n)$ which are easily proved.

- The functor

$$\begin{aligned} \Gamma_* \colon \text{quasi-coherent } \mathcal{O}_X\text{-mod} &\to \text{graded } S\text{-mod} \\ \mathcal{F} &\mapsto \bigoplus_{i \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes i}) \end{aligned}$$

is fully faithful, because $\mathcal{L}$ is ample. Hence we have an S-algebra isomorphism

$$\Lambda(n) \cong \mathrm{End}_S(\Gamma_*(\mathcal{E}_n)).$$

We shall describe the S-module $\Gamma_*(\mathcal{E}_n)$ in §6.

- $\Lambda(n)$ is a maximal order in $\Lambda(n) \otimes_S \mathrm{Frac}(S) \cong M_n(\mathrm{Frac}(S))$.
- The degree 0 part $\Lambda(n)_0 = \mathrm{End}(\mathcal{E}_n)$ is generated by a single endomorphism f defined by

$$f \colon \mathcal{E}_n \twoheadrightarrow \mathcal{E}_n/\mathcal{E}_1 \cong \mathcal{E}_{n-1} \hookrightarrow \mathcal{E}_n.$$

We have $f^n = 0$ and $\dim \Lambda(n)_0 = n$. We shall construct f explicitly in §7.

- The degree i part $\Lambda(n)_i$ has dimension $n^2 i$ for $i > 0$.

§4. Λ as an R-algebra

Write $\Lambda = \Lambda(n)$. Put $R = k[t, x]$, a polynomial subalgebra of S. Then $S = R \oplus Ry$. Λ is an R-free module of rank $2n^2$. We shall give an R-basis of Λ.

There exist $g \in \Lambda_1$, $h \in \Lambda_2$, $l \in \Lambda_3$ such that the following diagrams commute.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "g"] & \mathcal{E} \otimes \mathcal{L} \arrow[d] \\ \mathcal{O} \arrow[u] \arrow[r, "t"'] & \mathcal{L} \end{tikzcd}$$

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{h} & \mathcal{E}\otimes\mathcal{L}^{\otimes 2} \\
\uparrow & & \downarrow \\
\mathcal{O} & \xrightarrow[x]{} & \mathcal{L}^{\otimes 2}
\end{array}$$

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{l} & \mathcal{E}\otimes\mathcal{L}^{\otimes 3} \\
\uparrow & & \downarrow \\
\mathcal{O} & \xrightarrow[y]{} & \mathcal{L}^{\otimes 3}
\end{array}$$

Here the left vertical arrows are the inclusion map and the right ones are induced by the surjection $\mathcal{E} \to \mathcal{O}$. An explicit form of g will be given in §7. Then the following monomials form an R-basis of Λ.

$$f^i, f^igf^j, f^ihf^j, f^il \qquad 0 \le i \le n-1, \quad 0 \le j \le n-2.$$

The quotient $\bar{\Lambda} = \Lambda/R_+\Lambda = \Lambda/(t,x)\Lambda$ is a symmetric graded k-algebra of dimension $2n^2$. We have the following isomorphisms of bimodules over $\bar{\Lambda}_0 = \Lambda_0$.

$$\begin{aligned}
&\bar{\Lambda}_1 \cong \bar{\Lambda}_2 \cong \mathrm{Ker}(\bar{\Lambda}_0 \otimes \bar{\Lambda}_0 \xrightarrow{\text{mult}} \bar{\Lambda}_0) \\
&\bar{\Lambda}_3 \cong \bar{\Lambda}_0 \\
&\bar{\Lambda}_i = 0 \qquad i > 3.
\end{aligned}$$

§5. Λ as a k-algebra

Let $n > 2$. Regard Λ as a left $\Lambda_0 \otimes \Lambda_0$-module by $(a \otimes b) \cdot c = acb$.

Proposition. *$\Lambda_+ = \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ is a free $\Lambda_0 \otimes \Lambda_0$-module with basis*

$$(gf^{n-1})^i g, \quad (gf^{n-1})^i (gf^{n-2})^j gf^{n-3} g \quad \text{for } i, j \ge 0.$$

Theorem. *The k-algebra Λ is generated by f and g. The relations between them are generated by the following ones.*

Case n is even: $f^n = 0$ and $n-2$ quadratic relations of the form

$$\begin{aligned}
gf^kg =\, &A_k \cdot gf^{n-3}g + B_k \cdot gf^{n-1}g \\
&\text{with } A_k, B_k \in \Lambda_0 \otimes \Lambda_0 \text{ for } 0 \le k \le n-2,\ k \ne n-3
\end{aligned}$$

Case n is odd: $f^n = 0$ and $n-2$ quadratic relations as above and one cubic relation of the form

$$\begin{aligned}
gf^{n-3}gf^{n-3}g =\, &C \cdot gf^{n-2}gf^{n-3}g + D \cdot gf^{n-1}gf^{n-3}g + E \cdot gf^{n-1}gf^{n-1}g \\
&\text{with } C, D, E \in \Lambda_0 \otimes \Lambda_0.
\end{aligned}$$

§6. S-module $\Gamma_*(\mathcal{E}_n)$

Put $v = x - (\lambda+1)t^2$, $u = (x - t^2)(x - \lambda t^2)$. Define a graded S-module M as follows. M is R-free with basis $\alpha, \beta_i, \gamma_i$ for $i > 0$ with $\deg \alpha = 0$, $\deg \beta_i = 1$, $\deg \gamma_i = 2$. The action of y on M is given by

$$\begin{aligned} y\alpha &= x\beta_1 + t\gamma_1 \\ y\beta_i &= -\lambda t^3 O_i \beta_{i-1} - tx\beta_{i+1} + v\gamma_{i-1} - t^2\gamma_{i+1} \\ y\gamma_i &= x^2\beta_{i+1} + \lambda t^3 E_i \gamma_{i-1} + tx\gamma_{i+1} \end{aligned}$$

where $\beta_0 = -t\alpha$, $\gamma_0 = x\alpha$ and $O_i = 1$ for an odd i, $O_i = 0$ for an even i, $E_i = 1 - O_i$.

For $n \geq 1$ define a graded S-submodule $M(n)$ of M to be the free R-submodule generated by $\alpha, \beta_i, \gamma_i$ for $1 \leq i \leq n-1$ and $x\beta_n + t\gamma_n$.

PROPOSITION. *$\Gamma_*(\mathcal{E}_n) \cong M(n)$ as graded S-modules.*

So we may identify $\Lambda(n) = \mathrm{End}_S(M(n))$.

Though the S-module M is not free, the $S[\frac{1}{y}]$-module $M[\frac{1}{y}] = S[\frac{1}{y}] \otimes_S M$ is free with basis α_i, $i \geq 0$, given by

$$\begin{aligned} \alpha_i &= \frac{1}{x}\gamma_i && i\text{: odd} \\ &= -\frac{1}{u}(\lambda t^3 \beta_i - v\gamma_i) && i\text{: even.} \end{aligned}$$

§7. Generators

Let us construct $f, g \in \Lambda$ as endomorphisms of the S-module $M(n)$. Define an $S[\frac{1}{y}]$-linear map $f\colon M[\frac{1}{y}] \to M[\frac{1}{y}]$ by

$$\begin{aligned} f(\alpha_i) &= \alpha_{i-1} - \frac{\lambda t^3 y}{ux}\alpha_{i-2} + \frac{((\lambda+1)v + \lambda t^2)x}{u}\alpha_{i-3} \\ &\quad - \frac{\lambda t y}{u}\alpha_{i-4} + \frac{\lambda v x}{u}\alpha_{i-5} && \text{if } i \text{ is even} \\ f(\alpha_i) &= \alpha_{i-1} + \frac{\lambda t^3 y}{ux}\alpha_{i-2} \\ &\quad + \frac{(\lambda+1)x - \lambda t^2}{x}\alpha_{i-3} + \frac{\lambda t y}{u}\alpha_{i-4} && \text{if } i \text{ is odd} \end{aligned}$$

where we understand $\alpha_i = 0$ for $i < 0$. Then

$$\begin{aligned}
f(\alpha) &= 0 \\
f(\beta_i) &= \beta_{i-1} + (\lambda+1)\beta_{i-3} && i\text{: even} \\
&= \beta_{i-1} + (\lambda+1)\beta_{i-3} + \lambda\beta_{i-5} && i\text{: odd} \\
f(\gamma_i) &= \gamma_{i-1} + (\lambda+1)\gamma_{i-3} + \lambda\gamma_{i-5} - \lambda t\beta_{i-3} && i\text{: even} \\
&= \gamma_{i-1} + (\lambda+1)\gamma_{i-3} + \lambda t\beta_{i-3} && i\text{: odd}
\end{aligned}$$

So M and $M(n)$ are stable under f. We denote also by f the restrictions of f to M and $M(n)$. Thus $f \in \Lambda(n)_0$ for all n.

Secondly, define an $S[\frac{1}{y}]$-linear map $g\colon M[\frac{1}{y}] \to M(n)[\frac{1}{y}]$ as follows. When n is even,

$$\begin{aligned}
g(\alpha_0) &= t\alpha_{n-1} - \frac{y}{x}\alpha_{n-2} \\
g(\alpha_1) &= \frac{y}{x}\alpha_{n-1} + \frac{t((\lambda+1)x - \lambda t^2)}{x}\alpha_{n-2} + \frac{\lambda t^2 y}{u}\alpha_{n-3} \\
g(\alpha_2) &= -\frac{\lambda t^2 y}{u}\alpha_{n-2} + \frac{\lambda t v x}{u}\alpha_{n-3} \\
g(\alpha_i) &= 0 \quad \text{for } i > 2,
\end{aligned}$$

and when n is odd,

$$\begin{aligned}
g(\alpha_0) &= t\alpha_{n-1} - \frac{vy}{u}\alpha_{n-2} \\
g(\alpha_1) &= \frac{y}{x}\alpha_{n-1} + (\lambda+1)t\alpha_{n-2} \\
g(\alpha_2) &= -\frac{\lambda t^2 y}{u}\alpha_{n-2} + \sum_{i\geq 3,\mathrm{odd}} \lambda(-\lambda-1)^{(i-3)/2}\left(t\alpha_{n-i} - \frac{vy}{u}\alpha_{n-i-1}\right) \\
g(\alpha_i) &= 0 \quad \text{for } i > 2.
\end{aligned}$$

Then it turns out that g maps M into $M(n)$. Its restriction $M(n) \to M(n)$ is denoted by g again. g increases degree by 1, so $g \in \Lambda_1$. These f, g are the desired generators.

§8. Explicit equations in case n even

When n is even, we can give explicit defining equations for Λ, using additional generators. We define $e \in \Lambda_0$ and $g_+ \in \Lambda_1$ by

$$\begin{aligned}
e(\alpha_i) &= \alpha_{i-2} \quad \text{for all } i \\
g_+(\alpha_0) &= t\alpha_{n-2} - \frac{vy}{u}\alpha_{n-3} \\
g_+(\alpha_1) &= t\alpha_{n-1} + (\lambda+1)t\alpha_{n-3} \\
g_+(\alpha_2) &= \frac{vy}{u}\alpha_{n-1} + (\lambda+1)t\alpha_{n-2} \\
g_+(\alpha_i) &= 0 \quad \text{for } i > 2.
\end{aligned}$$

THEOREM. *If n is even and $n > 2$, the k-algebra Λ has the following presentation. The generators are f, e, g, g_+. The relations are*

$$\begin{aligned}
&e^{\frac{n}{2}} = 0 \\
&f^2 = (1+(\lambda+1)e)(1+\lambda e)(1+e)e \\
&fg(1+(\lambda+1)e) + (1+(\lambda+1)e)gf \\
&\quad = g_+ + (\lambda+1)eg_+ + (\lambda+1)g_+e + \lambda e^2 g_+ + ((\lambda+1)^2+\lambda)eg_+e + \lambda g_+e^2 \\
&\qquad + \lambda(\lambda+1)e^2 g_+ e + \lambda(\lambda+1)eg_+e^2 \\
&ge^{\frac{n-4}{2}}g = \lambda g_+ e^{\frac{n-2}{2}} g_+ \\
&g_+e^{\frac{n-4}{2}}g_+ = (\lambda+1)g_+e^{\frac{n-2}{2}}g_+ \\
&ge^j g = ge^j g_+ = 0 \quad \text{for } 0 \le j \le \tfrac{n-6}{2}.
\end{aligned}$$

Finally we give another presentation of Λ in line with the theorem of §5. Put

$$c = e \otimes 1, d = 1 \otimes e, p = f \otimes 1, q = 1 \otimes f \in \Lambda_0 \otimes \Lambda_0$$

and

$$\begin{aligned}
\alpha &= (1+(\lambda+1)c)(1+(\lambda+1)d) - \lambda^2 c^2 d^2 \\
\gamma &= (\lambda+1)(1+\lambda c)(1+c)(1+\lambda d)(1+d) \\
&\quad + \lambda d(1+\lambda c)(1+c) + \lambda c(1+\lambda d)(1+d) \\
\beta &= (1+\lambda cd)\alpha - (\lambda+1)cd\gamma \\
&= 1 + (\lambda+1)(c+d) + \lambda cd - (\lambda+1)^3(c^2 d + cd^2) \\
&\quad - \left((\lambda+1)^4 + \lambda(\lambda+1)^2 + \lambda^2\right)c^2 d^2 - \lambda(\lambda+1)^2(c^3 d + cd^3) \\
&\quad - \lambda(\lambda+1)((\lambda+1)^2+\lambda)(c^3 d^2 + c^2 d^3) - \lambda^2((\lambda+1)^2+\lambda)c^3 d^3.
\end{aligned}$$

Then $\alpha, \beta, \gamma \in \Lambda_0 \otimes \Lambda_0$ and β is invertible.

Theorem. *If n is even and $n > 2$, the k-algebra Λ has the following presentation. The generators are f, e, g. The relations are*

$$e^{\frac{n}{2}} = 0$$

$$f^2 = (1+(\lambda+1)e)(1+\lambda e)(1+e)e$$

$$ge^{\frac{n-2}{2}}g = (\square_1 p + \square_2 q)ge^{\frac{n-4}{2}}fg + (\square_3 p + \square_4 q)ge^{\frac{n-2}{2}}fg$$

$$\square_1 = -\frac{1}{\beta}(1+\lambda d)(1+d)(1+(\lambda+1)d+\lambda cd)$$

$$\begin{aligned}\square_3 = \frac{1}{\beta}(1+\lambda d)(1+d)\big[&(\lambda+1)(1+(\lambda+1)d) \\ &+(\lambda+1+\frac{\lambda c}{(1+\lambda c)(1+c)})(1+(\lambda+1)d+\lambda cd)\big]\end{aligned}$$

$\square_1 \leftrightarrow \square_2, \quad \square_3 \leftrightarrow \square_4$ *by interchange* $c \leftrightarrow d$

$$ge^{\frac{n-4}{2}}g = (\square_1 p + \square_2 q)ge^{\frac{n-4}{2}}fg + (\square_3 p + \square_4 q)ge^{\frac{n-2}{2}}fg$$

$$\square_1 = -\frac{1}{\beta}d(1+(\lambda+1)d)(1+(\lambda+1)c+\lambda cd)$$

$$\begin{aligned}\square_3 = \frac{1}{\beta}(1+(\lambda+1)d)\big[&(\lambda+1)d(1+(\lambda+1)c+\lambda cd) \\ &+\frac{1+(\lambda+1)c}{(1+\lambda c)(1+c)}(1+(\lambda+1)d+\lambda cd)\big]\end{aligned}$$

$\square_1 \leftrightarrow \square_2, \quad \square_3 \leftrightarrow \square_4$ *by interchange* $c \leftrightarrow d$

$$ge^{\frac{n-k}{2}}g = 0 \quad \text{for } k > 4, \text{ even}$$

$$ge^{\frac{n-6}{2}}fg = (\square_1 + \square_2 pq)ge^{\frac{n-4}{2}}fg + (\square_3 + \square_4 pq)ge^{\frac{n-2}{2}}fg$$

$$\square_1 = \frac{1}{\beta}((\lambda+1)\beta - \lambda\gamma cd)$$

$$\square_2 = -\frac{1}{\beta}\lambda(1+\lambda cd)$$

$$\begin{aligned}\square_3 = \frac{1}{\beta}\big[&\lambda(1+\lambda cd)(1+(\lambda+1)c)(1+(\lambda+1)d) \\ &-(\lambda+1)^2\beta + \lambda(\lambda+1)\gamma cd\big]\end{aligned}$$

$$\square_4 = \frac{1}{\beta}(\frac{\lambda\gamma}{(1+\lambda c)(1+c)(1+\lambda d)(1+d)} + \lambda(\lambda+1)(1+\lambda cd))$$

$$ge^{\frac{n-8}{2}}fg = (\square_1 + \square_2 pq)ge^{\frac{n-4}{2}}fg + (\square_3 + \square_4 pq)ge^{\frac{n-2}{2}}fg$$

$$\square_1 = \frac{1}{\beta}(1+(\lambda+1)c)(1+(\lambda+1)d)$$
$$\times\left(1-(\lambda+1)^2cd-\lambda(\lambda+1)(c^2d+cd^2)-\lambda^2c^2d^2\right)$$
$$\square_2 = -\frac{1}{\beta}\lambda(\lambda+1)cd$$
$$\square_3 = -\frac{1}{\beta}(\lambda+1)(1+(\lambda+1)c)(1+(\lambda+1)d)$$
$$\times\left(1-((\lambda+1)^2+\lambda)cd-\lambda(\lambda+1)(c^2d+cd^2)-\lambda^2c^2d^2)\right)$$
$$\square_4 = \frac{1}{\beta}\left(\frac{\lambda\alpha}{(1+\lambda c)(1+c)(1+\lambda d)(1+d)}+\lambda(\lambda+1)^2cd\right)$$

$ge^{\frac{n-k}{2}}fg = 0$ *for* $k > 8$, *even.*

References

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[3] D. Tambara, Graded algebras of vector bundle maps over an elliptic curve, preprint.